

On canonical pairs in two-transverse-mode DOPOs

Carlos Navarrete-Benlloch, Eugenio Roldán, and Germán J. de Valcárcel

Departament d'Òptica, Universitat de València, Dr. Moliner 50, 46100-Burjassot, Spain.

Abstract

In ref. [1] we analyzed the properties of a Degenerate Optical Parametric Oscillator (DOPO) tuned to the first transverse mode family at the signal frequency. Above threshold, a Hermite-Gauss mode with an arbitrary orientation in the transverse plane is emitted, and thus the rotational invariance of the system is broken. When quantum effects were taken into account, it was found on the one hand, that quantum noise is able to induce a random rotation on this classically emitted mode. On the other hand, the analysis of a balanced homodyne detection in which the local oscillator (LO) was orthogonal to the excited mode at any time, showed that squeezing in the quadrature selected by the LO was found for every phase ψ_L of this one, squeezing being perfect for $\psi_L = \pi/2$. This last fact revealed an apparent paradox: If all quadratures are below shot noise level, the uncertainty principle seems to be violated. In [1] we stated that the explanation behind this paradox is that the quadratures of the rotating orthogonal mode do not form a canonical pair, and the extra noise is transferred to the diffusing orientation. These notes are devoted to prove this claim.

PACS numbers: 42.50.L; 42.65.Sf

Statement of the problem. In the work of Ref. [1] we showed the possibility of using the spontaneous rotational symmetry breaking that occurs in the transverse plane of some optical systems to produce non-critically squeezed light. In particular, we proved this in the case of a type I Degenerate Optical Parametric Oscillator (DOPO) tuned to the first transverse mode family at the signal frequency. Let us remind what was found there. [If you are familiar with the results showed in [1] just jump to Eq. (7).]

We consider a type I DOPO pumped by a Gaussian resonant beam at frequency $2\omega_0$. At the subharmonic, i.e., at ω_0 , the cavity is assumed to be tuned to the first transverse mode family, which supports two Laguerre-Gauss modes, $L_{\pm 1}(\mathbf{r})$, with opposite orbital angular momenta: $L_{\pm 1}(\mathbf{r}) = \pi^{-1/2} w^{-2} r e^{-r^2/2w^2} e^{\pm i\phi}$, where r and ϕ are the polar coordinates in the transverse plane, and $\sqrt{2}w$ is the waist radius of the signal beam.

The signal field operator at frequency ω_0 can then be written as

$$\hat{E}_s(\mathbf{r}, t) = \hat{A}_s(\mathbf{r}, t) e^{-i\omega_0 t} + \text{H.c.}, \quad (1)$$

apart from an unimportant constant factor, where the slowly varying envelope

$$\hat{A}_s(\mathbf{r}, t) = \hat{a}_{+1}(t) L_{+1}(\mathbf{r}) + \hat{a}_{-1}(t) L_{-1}(\mathbf{r}) \quad (2)$$

and the interaction picture boson operators satisfy the usual canonical commutation relations

$$[\hat{a}_i(t), \hat{a}_j^\dagger(t)] = \delta_{ij} \quad i, j = \pm 1. \quad (3)$$

In the classical limit, the modal boson operators $(\hat{a}_j, \hat{a}_j^\dagger)$ coincide with the normal variables for each mode (α_j, α_j^*) , and the first result we proved in [1] was that the long term classical emission of the DOPO pumped above threshold for signal modes oscillation is given by

$$\bar{\alpha}_{\pm 1} = \rho e^{\mp i\theta} \quad (4)$$

where ρ is an amplitude which depends on the system parameters and whose exact dependence is not important for these notes, and θ is an *arbitrary* phase, i.e., the phase difference between the $\alpha_{\pm 1}$ modes is not fixed by the classical equations of the system.

When the quantum properties of the system are analyzed, we take this classically undefined phase θ as a quantum variable in the positive P -representation, finding that it diffuses in an undamped way because of quantum noise. Hence, under these circumstances the emission of the system will be given by (we omit the \mathbf{r} dependences of the modes)

$$A_s = \rho \left[e^{-i\theta(t)} L_{+1} + e^{i\theta(t)} L_{-1} \right] \propto r e^{-r^2/2w^2} \cos[\phi - \theta(t)] \quad (5)$$

i.e., the system is emitting in a Hermite-Gauss TEM₁₀ mode rotated by an angle $\theta(t)$ with respect to the x -axis. The random diffusion of θ means that the orientation of the classically excited mode is totally undetermined in the long time limit.

What was exploited in Ref. [1] is the following idea. If the angular orientation of the TEM₁₀ mode in the transverse plane is completely undetermined, it would be natural that its associated orbital angular momentum were completely determined, i.e., squeezed. This simple idea was proved rigorously in [1] by introducing a homodyne detection scheme in which, using a 50/50 beam splitter, the field exiting the DOPO is mixed with a local oscillator (LO) proportional to the orbital angular momentum of the classically excited mode $(-i\partial_\theta A_s)$, i.e.,

$$A_{LO} = \rho_L e^{i\psi_L} \left[e^{-i\theta(t)} L_{+1} - e^{i\theta(t)} L_{-1} \right] \propto \sin[\phi - \theta(t)] \quad (6)$$

which in this case coincides with another Hermite-Gauss mode, orthogonal to the classically emitted one at any time. Note that ρ_L is a real amplitude and we have denoted by ψ_L the phase of the LO. The best way to observe squeezing is to measure the spectrum of the intensity dif-

ference between the two output ports of the beam splitter, denoted by $V(\omega)$. If $V(\omega_*) = 0$, one can say that the field generated by the DOPO at noise frequency ω_* has no noise in the quadrature selected by the LO, i.e. it is perfectly squeezed.

When the calculation of this spectrum is carried out one obtains [1]

$$V_{\psi_L}(\omega) = 1 - \frac{\sin^2(\psi_L)}{1 + (\omega/2\gamma_s)^2} \quad (7)$$

where γ_s is the cavity linewidth at the signal frequency. This result confirms what we suspected with the simple reasoning given some lines above: The phase quadrature ($\psi_L = \pi/2$) of the angular momentum of the classically excited mode is perfectly squeezed at the signal frequency ($\omega = 0$). Moreover for any LO phase $V_{\psi_L}(\omega) \leq 1$, i.e., any quadrature of the selected mode has noise reduction (except $\psi_L = 0$ where the equality of this expression holds). This is an unexpected result, as in usual DOPOs when noise is removed from one quadrature, it is transferred to its orthogonal quadrature in order to fulfil the uncertainty principle, as they form a canonical pair.

In [1] we stated that this result can be understood if the two orthogonal *detected* quadratures do not form a canonical pair, and that this would be because the squeezed mode (and hence the LO) is rotating randomly (because of the already commented diffusion of θ). Hence, the noise suppression of any rotating quadrature is not transferred to another quadrature, but to the orientation of the squeezed mode. If so, the undefined orientation and the squeezed quadrature have to form the canonical pair. The rest of the notes are devoted to prove that this intuitive explanation is actually correct.

The way up. The proof that reinforces the previous explanation relies on the quantum operator which is detected by the scheme we presented some lines above, i.e., the *rotating* quadrature. By projecting the field exiting the cavity onto the LO one can find that this operator is (we have reversed the coherent representation for the θ variable)

$$\hat{X}^{\psi_L} = \frac{i}{\sqrt{2}} \left[e^{-i\psi_L} \left(e^{i\hat{\theta}} \hat{a}_{+1} - e^{-i\hat{\theta}} \hat{a}_{-1} \right) \right] + \text{H.c.} \quad (8)$$

where $\hat{\theta}$ is half the phase difference operator [2] between opposite angular momentum modes whose exponential form is given in terms of the boson operators by [3]

$$\hat{U} = \exp(i\hat{\theta}) = \left[\hat{U}_{+1}^\dagger \hat{U}_{-1} + \sum_{n=0}^{\infty} |0, n\rangle \langle n, 0| e^{i\phi(n)} \right]^{1/2} \quad (9)$$

being

$$\hat{U}_j = \frac{1}{\sqrt{\hat{a}_j^\dagger \hat{a}_j + 1}} \hat{a}_j \quad (10)$$

the Susskind-Glogower phase operator of the mode j , the state $|m, n\rangle = |m\rangle_{+1} \otimes |n\rangle_{-1}$ a vector of the number state basis for the joined Hilbert space of both modes ($\langle m, n| = \langle m|_{+1} \otimes \langle n|_{-1}$) and $\phi(n)$ an arbitrary function defined on the natural numbers. For reasons that should be clear from the Introduction above, we will call $\hat{\theta}$ the *orientation* operator, which is simply given by

$$\hat{\theta} = \frac{1}{i} \ln \hat{U}. \quad (11)$$

Now the way to follow seems clear; we want to prove that \hat{X}^{ψ_L} and $\hat{X}^{\psi_L + \frac{\pi}{2}}$ do not form a canonical pair, while \hat{X}^{ψ_L} and $\hat{\theta}$ do.

On the other hand, two operators \hat{F} and \hat{G} are canonically related if they satisfy a commutation relation of the kind

$$[\hat{F}, \hat{G}] = iC \quad (12)$$

where C is a real non-zero number ($C = 2$ for usual orthogonal quadratures \hat{X}^φ and $\hat{X}^{\varphi + \frac{\pi}{2}}$). Hence, if we can prove that

$$[\hat{X}^{\psi_L}, \hat{X}^{\psi_L + \frac{\pi}{2}}] = 0 \text{ and } [\hat{X}^{\psi_L}, \hat{\theta}] = iC \quad (13)$$

we will give the proof we are searching for.

However, there is one easier way to prove if two operators are canonically related or not: instead of using Quantum Field Theory (QFT) one can just prove whether two observables are canonically related or not via Poisson brackets in a Classical Field Theory (CFT) context. This idea is the one we develop in the next part.

The classical field theory resort. The usual approach one uses to move from CFT to QFT is to change the classical normal variables for each mode of the field, α_j and α_j^* , by boson operators \hat{a}_j and \hat{a}_j^\dagger , satisfying commutation relations

$$[\hat{a}_j, \hat{a}_j^\dagger] = i \{ \alpha_j, \alpha_j^* \} \quad (14)$$

where $\{F, G\}$ denotes the poisson bracket between two functions $F(\alpha_j, \alpha_j^*)$ and $G(\alpha_j, \alpha_j^*)$ defined as

$$\{F, G\} = \frac{1}{i} \sum_j \frac{\partial F}{\partial \alpha_j} \frac{\partial G}{\partial \alpha_j^*} - \frac{\partial F}{\partial \alpha_j^*} \frac{\partial G}{\partial \alpha_j}. \quad (15)$$

It can be checked out that by applying this definition onto the fundamental brackets $\{ \alpha_j, \alpha_j^* \}$ makes (14) coincide with the canonical commutation relations (3).

The Poisson bracket of two monomode orthogonal quadratures $X_j = \alpha_j + \alpha_j^*$ and $Y_j = -i(\alpha_j - \alpha_j^*)$ is then

$$\{X_j, Y_j\} = 2, \quad (16)$$

as expected by the commutation relation (12). In general two classical functions $F(\alpha_j, \alpha_j^*)$ and $G(\alpha_j, \alpha_j^*)$ are said

to form a canonical pair if their Poisson bracket is of the kind

$$\{F, G\} = C \quad (17)$$

where C is again a real number. Hence, we don't need to calculate the difficult commutators (13) to prove what we want, we can just compute the analogous Poisson brackets (which only need making some derivatives) for functions defined in a context of CFT.

In particular, the functions we are interested in, are the classical counterparts of (8) and (11) which are given by

$$X^{\psi_L} = \frac{i}{\sqrt{2}} [e^{-i\psi_L} (e^{i\theta} \alpha_{+1} - e^{-i\theta} \alpha_{-1})] + \text{c.c.} \quad (18)$$

with

$$e^{i\theta} = \frac{\alpha_{+1}^* \alpha_{-1}}{\sqrt{\alpha_{+1}^* \alpha_{+1}} \sqrt{\alpha_{-1}^* \alpha_{-1}}}, \quad (19)$$

and

$$\theta = \frac{1}{i} \ln \left[\frac{\alpha_{+1}^* \alpha_{-1}}{\sqrt{\alpha_{+1}^* \alpha_{+1}} \sqrt{\alpha_{-1}^* \alpha_{-1}}} \right]. \quad (20)$$

If we can prove that

$$\{X^{\psi_L}, X^{\psi_L + \frac{\pi}{2}}\} = 0 \text{ and } \{X^{\psi_L}, \theta\} = C \quad (21)$$

we will prove what we are trying too.

Using the definition of the Poisson brackets and after some algebra, it is possible to show that

$$\{X^{\psi_L}, X^{\psi_L + \frac{\pi}{2}}\} = \frac{|\alpha_{-1}| - |\alpha_{+1}|}{2 |\alpha_{+1}| |\alpha_{-1}|} \quad (22)$$

and

$$\{X^{\psi_L}, \theta\} = \frac{i (|\alpha_{+1}| + |\alpha_{-1}|) (e^{i\psi_L} |\alpha_{-1}| - e^{-i\psi_L} |\alpha_{-1}|)}{4\sqrt{2} |\alpha_{+1}|^{3/2} |\alpha_{-1}|^{3/2}} \quad (23)$$

with $|\alpha_j| = \sqrt{\alpha_j^* \alpha_j}$.

On the other hand, in the DOPO that was treated in [1], the number of photons with opposite angular momentum is sensibly equal, i.e., $|\alpha_{+1}| \approx |\alpha_{-1}|$; hence, the dominant term of the previous brackets will be that with $|\alpha_{+1}| = |\alpha_{-1}| = \rho$, and thus

$$\{X^{\psi_L}, X^{\psi_L + \frac{\pi}{2}}\} \approx 0 \quad (24)$$

and

$$\{X^{\psi_L}, \theta\} \approx -\frac{\sin \psi_L}{\sqrt{2}\rho}. \quad (25)$$

By comparing this with (21) we find that this is a confirmation of what we expected: the classical field variables X^{ψ_L} and $X^{\psi_L + \frac{\pi}{2}}$ do not form a canonical pair

(moreover, they commute), while X^{ψ_L} and θ do. This can be seen as an indirect proof of the same conclusion for the operators \hat{X}^{ψ_L} , $\hat{X}^{\psi_L + \frac{\pi}{2}}$ and $\hat{\theta}$.

Conclusions. In these notes we have proven that the apparent violation of the uncertainty principle that shows the spectrum (7) obtained in [1] is just that: apparent. Two detected orthogonal quadratures are not canonically related as the orientation diffusion has been removed from the experiment by transferring it to the local oscillator field. Hence, quadratures can be squeezed without interchanging noise between each other as all the extra noise is carried by the orientation angle θ , which is the canonically related variable of all the squeezed quadratures. We have given here a simple proof based on classical field theory calculus.

-
- [1] C. Navarrete-Benlloch, E. Roldán and G. J. de Valcárcel, arXiv: 0709.0212v4 [quant-ph]
 - [2] A. Luis and L. L. Sánchez-Soto, Phys. Rev. A **48**, 4702 (1993)
 - [3] Sixia Yu, Phys. Rev. Lett. **79**, 780 (1997)